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THE WORD OF THE MUSES
(Plato, *Rep.* 8.546)

Ever since Proclus wrote his commentary on Plato's *Republic*, repeated attempts have been made to find a hidden number of cosmic significance in *Rep.* 8.546. For the Neo-Platonist it was natural to look for esoteric secrets in ancient works; among the men of the New Learning at the end of the Middle Ages there were enough astrologers and necromancers to ensure respect for the proposition; we are now again enamoured of irrationality. But the scholars who attempted such calculations around 1900 must have considered Plato himself a mystery-monger.

In this article I propose: (i) to show why such attempts are mistaken, (ii) to discuss what early writers who mention the passage say about its meaning, (iii) to provide a mathematician's translation that fits the context, and to comment on it; for the currently accepted explanation is unsatisfactory.

'There is fairly widespread agreement that the geometrical number is $12{,}960{,}000 = 3{,}600^2 = 4{,}800 \times 2{,}700$, but on the method by which this number is reached the widest divergence exists'[1] or, from an earlier, different guess: '...one can, so to speak, state *a priori* that Plato's number is a multiple of 19 ten thousands',[2] i.e. the text is approached with a ready-made answer in mind. There are three further objections to the conventional view. (*a*) It takes the text out of its context as if it had strayed from the *Timaeus*. M. Denkinger even speaks of 'the enigma of the number which Plato had the whim to insert among the first pages of book 8 of the *Republic* (546b–c)', and repeats 'it is inserted' in his appendices.[3] (*b*) The solution necessitates one impossible translation and usually involves a second one. On these, below. (*c*) It is concerned with no more than the last seventeen words of the whole text, of which the chief part is all too often left as an undigested lump.

All Plato's mathematical passages must be understood in their context. They are of course of interest for the history of mathematics, but Plato did not teach the subject, let alone provide a system of it; his hearers may be supposed to have been competent mathematicians who needed no more than hints at what he was referring to. Such passages as a rule illuminate a particular point in the argument; they are illustrations – a literary use of mathematics, one might say.

At the beginning of *Republic* 8 the context requires not the revelation of a mystic number but an explanation of why the leaders should be bound to go wrong. The explanation is given in the form of a mathematical process which can be compared to the leaders' task of finding the most suitable pairs of spouses. There was no secret to discover; there was a process described which Plato's contemporaries understood well enough, but which was no longer familiar to later generations.

It would seem that the knowledge waned gradually. Cicero[4] describes a problem as

[1] I. Thomas, *Greek Mathematical Works* I (Loeb), 400n.; cf. R. G. Bury in *CR* 33 (1919), 45–6 on A. G. Laird, *Plato's Geometrical Number and the Comment of Proclus*. 'While accepting Adam's solution...he maintains that "his method of reaching the 3600^2 is wrong..." ...On Mr Laird's view, the number 216, on which Adam set such store, seems to disappear, and with it, apparently, much of the pertinence of the whole passage to the subject of "better and worse births".'

[2] J. Dupuis, *Théon de Smyrne* (Paris, 1892), 372n.

[3] M. Denkinger, *REG* 68 (1955), 38, 71.

[4] *Ad Att.* 7.13.5.

'obscurer than Plato's number', but that does not mean that Plato was no longer understood in his time: Cicero after all goes on to give an explanation of his own problem. He just does not hold with this kind of roundabout argument. As Quintilian said (particularly on the *Timaeus*), it 'cannot be understood except by those who have diligently possessed themselves of this part of learning'.[5]

But those who prided themselves on 'this part of learning', the Neo-Platonists, were interested not in mathematics but in the hidden powers and esoteric meanings they extracted from the texts, till by Proclus' time the knowledge had vanished and the very word 'mathematician' had acquired a sinister meaning, as the uninitiated naturally distrusted men who claimed to have secret powers.

Regarding *Republic* 8.546, Proclus speaks of 'the hidden and dark way which is proper for solemn things' and says Plato 'did not want the holy to be readily available to the masses'.[6]

The Theodosian Code in the same century treated 'malefici, mathematici et ceteri similes' together, with the interpretation 'evildoers or spellbinders or senders of storms or those who stir up the minds of men by calling on demons', and the Lex Dei includes a title 'De mathematicis, maleficis et Manichaeis'.[7] 'Mathematicians' were by then just sorcerers, and 'Anyone who consults mathematicians, soothsayers, haruspices, prophets about the welfare of the *princeps* or the supreme power of the state receives the death sentence, together with anyone who answers'.[8] This branch of mathematics, astrology – tacked on to astronomy –, had long been considered suspect, though to Cicero, for example, mathematics was an honourable pursuit, like philosophy, oratory and poetry.[9] The 'mathematicians' who were expelled from Italy under the early emperors[10] were those whom Juvenal[11] equates with Chaldeans, astrologers, and haruspices, and whom the authorities considered not only fraudulent but politically dangerous.

It should be noted that Proclus calls the first part of his book on the *Republic* 'Commentary' (*peri*), whereas the second, which deals with our passage, is entitled 'Honeygatherer into the Word of the Muses'.[12] It is not a commentary giving rational explanations, but a compilation of texts and connecting remarks under several headings. 'It is necessary to have studied the geometrical number arithmetically as well as geometrically and also musically, if we can, and astronomically, then dialectically, rising into the human cycle and into the contemplation of the whole cosmos.'[13] And indeed, Proclus darts about, culling from what must have been an excellent library anything that appeared to him to have some bearing on the text. The method is quite modern: 'thinking' by association, not logically. The result is modern too: it would now be called 'enlarging the consciousness'. It may 'add several dimensions' to Plato's thought; it is not suitable for solving a mathematical problem. J. A. Adam said of this work:

It is disappointing to find that Proclus does not explain the meaning of the words but wanders off into astrological and other vagaries; 'freilich findet sich' says Hultsch 'bei Proclus keine

[5] *Inst. Orat.* 1.3.13.

[6] *In Plat. Rep.*, ed. W. Kroll, 2.8.4f.

[7] *Cod. Theod.* 9.13; cf. 16.25; 62. *Coll. Leg. Mos. et Rom. Tit.* 15 (in Riccobono, *FIRA* 2.578).

[8] *Pauli Sent.* 5.21.3 (*FIRA* 2.407).

[9] *De Orat.* 1.10.

[10] Tac. *Hist.* 1.22.

[11] 6.562; cf. Suet. *Vitell.* 14.3.

[12] *In Plat. Rep.* 2.1.3.

[13] Ibid. p. 36.

direkte Erläuterung der dunklen und vieldeutigen Worte Platon's; allein immerhin ist es ein Gewinn zu betrachten, dass die meisten der von Platon gebrauchten schwierigen Ausdrücke von dem Neuplatoniker wiederholt und in den Bereich seiner eigenen Spekulation aufgenommen worden sind.' It is clear that Proclus either would not or could not explain the passage: when he commits himself to a definite view, he is for the most part demonstrably wrong.[14]

Proclus[15] seems to have been the first to connect the marriage number with a very elegant method of approaching $\sqrt{2}$ by a sequence of isosceles triangles with rational sides, which is found in several ancient *Arithmetics*, e.g. Theon of Smyrna,[16] from where J. Dupuis[17] seems to have taken it directly. It has – *pace* F. Hultsch[18] – nothing to do with our passage;[19] the marriage number is not mentioned, and the terminology is entirely different, except for the expressions *διαμέτρων ῥητῶν* (Plato) and *ῥηταὶ διάμετροι* (Theon). Also 3 and 5 occur in both calculations – as they do in innumerable others – but in Plato 3 is a side, 5 the hypotenuse of the same triangle, whereas in the other text 5 is the side, and 3 the base of two different triangles; moreover, Theon's triangles are isosceles and not right-angled, Plato's right-angled and scalene. It is easy to see why the 'rational diameters' reminded the well read Proclus of this approximation, but we should consider seriously whether it could already have been formulated in Plato's time. (For a description of the method see Appendix.) The believer in the cosmic number extracts from a combination of the texts the desired factor, 48.

F. Hultsch did his very best for Proclus' reputation – and indeed Proclus' excerpts from works since lost are of the greatest importance – but he was forced to conclude: 'In general we find in Proclus from 37.12, instead of a methodical interpretation closely following Plato's text, no more than subjective combinations'.[20] Not just from 37.12 but throughout.

This is Burnet's text of the passage, with enough of the context to indicate its purpose (Oxford Classical Text, 545d7–546d3). The description of the 'number' extends from 546b4 *ἀνθρωπείῳ δέ* to *κύβων τριάδος*.

545 *ἦ βούλει, ὥσπερ*
Ὅμηρος, εὐχώμεθα ταῖς Μούσαις εἰπεῖν ἡμῖν ὅπως δὴ πρῶτον στάσις ἔμπεσε, καὶ φῶμεν αὐτὰς τραγικῶς ὡς πρὸς παῖδας ἡμᾶς παιζούσας καὶ ἐρεσχηλούσας, ὡς δὴ σπουδῇ λεγούσας, ὑψηλολογουμένας λέγειν;

546 *ἐπεὶ γενομένῳ παντὶ φθορά ἐστιν, οὐδ' ἡ τοιαύτη σύστασις τὸν ἅπαντα μενεῖ χρόνον, ἀλλὰ λυθήσεται.*

γένους δὲ ὑμετέρου εὐγονίας τε καὶ ἀφορίας, καίπερ ὄντες σοφοί,
b *οὓς ἡγεμόνας πόλεως ἐπαιδεύσασθε, οὐδὲν μᾶλλον λογισμῷ μετ' αἰσθήσεως τεύξονται, ἀλλὰ πάρεισιν αὐτοὺς καὶ γεννήσουσι παῖδάς ποτε οὐ δέον. ἔστι δὲ θείῳ μὲν γεννητῷ*

[14] James Adam, *The Nuptial Number of Plato* (London, 1891), 11 n. 1.
[15] *In Plat. Rep.* 2.26, 16f.
[16] Ed. Hiller, 43f.
[17] *Théon de Smyrne* 380.
[18] *Proclus, In Plat. Rep.* ed. Kroll, 2.394f.
[19] Cf. T. L. Heath in *The Legacy of Greece*, ed. R. W. Livingston (1922), 110.
[20] Proclus 2.405.

> περίοδος ἣν ἀριθμὸς περιλαμβάνει τέλειος, ἀνθρωπείῳ δὲ
> ἐν ᾧ πρώτῳ αὐξήσεις δυνάμεναί τε καὶ δυναστευόμεναι, τρεῖς
> ἀποστάσεις, τέτταρας δὲ ὅρους λαβοῦσαι ὁμοιούντων τε καὶ
> ἀνομοιούντων καὶ αὐξόντων καὶ φθινόντων, πάντα προσήγορα
> καὶ ῥητὰ πρὸς ἄλληλα ἀπέφηναν· ὧν ἐπίτριτος πυθμὴν c
> πεμπάδι συζυγεὶς δύο ἁρμονίας παρέχεται τρὶς αὐξηθείς,
> τὴν μὲν ἴσην ἰσάκις, ἑκατὸν τοσαυτάκις, τὴν δὲ ἰσομήκη
> μὲν τῇ, προμήκη δέ, ἑκατὸν μὲν ἀριθμῶν ἀπὸ διαμέτρων
> ῥητῶν πεμπάδος, δεομένων ἑνὸς ἑκάστων, ἀρρήτων δὲ δυοῖν,
> ἑκατὸν δὲ κύβων τριάδος. σύμπας δὲ οὗτος ἀριθμὸς γεω-
> μετρικός, τοιούτου κύριος, ἀμεινόνων τε καὶ χειρόνων γε-
> νέσεων, ἃς ὅταν ἀγνοήσαντες ὑμῖν οἱ φύλακες συνοικίζωσιν d
> νύμφας νυμφίοις παρὰ καιρόν, οὐκ εὐφυεῖς οὐδ' εὐτυχεῖς
> παῖδες ἔσονται·

The early writers who mention this passage in Plato never even hint at a cosmic number. Victor Cousin[21] stresses that the text was not obscure to them: 'they quote it, criticize it, comment on it, without seeming not to understand it'. Aristotle, accepting that the emergence of people whom no education can make excellent could cause the decay of the state, observes that this would apply to all constitutions.[22] Thus he confirms that the text deals with only one of a number of constitutions, so that a number spanning the whole world age is very unlikely to be involved. He calls the scheme a diagram; Plutarch calls it a nuptial plan.[23] Most commentators, down to Proclus (which shows that the information was traditional), refer to triangles.[24] These triangles have disappeared without trace in modern translations.

One should start with the earliest commentators, who are the most likely to know what they are talking about. Aristotle[25] indicates that Plato's argument has two parts (J. Adam's translation[26]): 'for he says that the fact that nothing abides but everything changes within a certain period, is the cause, while *the beginning* (sc. of the change) is of (i.e. belongs, comes from) those (i.e. the αὐξήσεις); the 3:4 base of which yoked to 5, furnishes two harmonies – meaning, when the number of this diagram is made solid'. 'Sc. of the overthrow' would be better than 'sc. of the change'.

Plutarch describes the handiman's triangle (stood on one of its sides); 3 the perpendicular, 4 the base, 5 the hypotenuse.[27] His interest is in the mysticism of numbers: 4 and 3 represent the divine pair Isis and Osiris.

Aristides Quintilianus[28] says: 'this is the first of all rational triangles we form (for the right-angled triangles from shorter sides evidently have one irrational side...); for this reason the 5 is also said to be the first rational diameter'. He and others find the

[21] *Oeuvres de Platon*, traduites par V. Cousin, quoted from Dupuis, op. cit. 370.

[22] *Pol.* 5.1316a.

[23] *De Is. et Os.* 373f. Oddly enough in view of this and of Plato's own words (*Rep.* 546c fin.) J. Dupuis objects to the appellation 'marriage number' (op. cit. 388).

[24] E.g. Aristides Quintilianus 3.23; Iamblichus, *Vita Pyth.* 27.130; Proclus, *In Eucl.* (ed. Friedlein) 427f.

[25] Loc. cit.

[26] *The Nuptial Number of Plato*, 23.

[27] Loc. cit.

[28] Loc. cit.; also Alex. Aphrod. *In Metaph.* 56.19f. (Hayduck 75.27f.).

most unexpected (Pythagorean) information in this triangle. But he also states that this basic figure contains four proportions, as the sides can be cut thus:

the 3 into 2:1, the diapason,
the 4 into equal parts, 2:2 = 1:1, the prime,
the 5 into 3:2, the ἡμιόλιον,

and the sides enclosing the right angle give 4:3, the epitriton. He therefore finds four harmonies to Plato's two. And he adds 'Plato said of this, the basis of 4:3 yoked together by the 5', which proves that Aristides is explaining *Rep.* 8.546.

Iamblichus – quoting Plato's description – considers the basic Pythagorean triangle the best image of the State,[29] which shows that he no longer understood the passage. This was in the fourth century A.D.

Proclus mentions Plato and this passage several times in his commentary on Euclid, giving among other things traditional wisdom. He says correctly that in the triangle in the *Republic* the 3 and the 4 enclose the right angle, i.e. that they are sides of a triangle, not a part of a weird sum.

Other authors must also be consulted to establish the meaning of some technical mathematical terms, but the two mistranslations common nowadays can easily be corrected from the writings of Plato and Aristotle themselves.

The first mistranslated term is Plato's ἀρρήτων. Plato does not use any of the words that may mean 'irrational' in this technical sense, except in *Hippias Maior* 303b, where the use of ἄρρητος should be seen as a further indication that the work is spurious. ἀρρήτων must be translated 'that must not be said' or 'that cannot be expressed'. The meaning 'irrational' seems to be exceedingly rare: apart from the *Hippias Maior*, I have only found it in Athenagoras, Aristides Quintilianus[30] and Proclus. 'Not commensurable' (οὐ σύμμετρος) is found twice in Plato, in *Theaet.* 147d and 148b.

An example of the second mistranslation is found in Adam's interpretation of the Aristotle passage, which is given above in his translation, 'make it solid, as Aristotle bids us, i.e. cube it'.[31] But *stereos* does not mean 'cubic', certainly not in Aristotle, who uses κυβικοὶ ἀριθμοί.[32] According to Bonitz,[33] Aristotle uses *stereos* (*a*) in the ordinary sense 'hard', 'firm', in the works on natural science, and once as 'fixed': 'the two points remain fixed';[34] (*b*) in the mathematical sense of 'solid figure', most often in the time-honoured phrase 'point, line, plane, solid figure'.[35] Finally (*c*) there is a specifically philosophical sense: 'Democritus has *to stereon kai kenon*, of which he speaks as the being and the not being...',[36] 'the mental picture of the flat number, the perception of the solid'.[37] The contrast between mathematical, purely noetic perceptions, and the actual things which can be perceived by the senses, appears frequently, but without using the word *stereos*.[38] Philo in his praise of the number four explains the term in a way modern persons can follow: 'the four has led us out of the bodiless purely mental realm into the conception of the three-dimensional body, the first in nature to be perceptible to the senses'.[39] It is the four which brings us into

[29] Loc. cit. [30] Diels, *Vorsokrat.* 1.421.5; Arist. Quint., loc. cit.
[31] *The Nuptial Number of Plato*, 24. [32] *Probl.* 15.3, 910b36. [33] *Index Aristotelicus.*
[34] *De mundio* 2.391b 19, prob. spurious.
[35] In the same sense in *De caelo* 2.4, 286b13; 3.5, 304a15; 306b7.
[36] *Physica* 1.5, 188a22f. [37] *De Anim.* 1.2, 404b23f.
[38] E.g. *Metaph.* 1.21, 8, 15f.; 24, 8, 30f.; Plato, *Rep.* 534c; Philo, *De opif. mundi* 92 (Cohn-Wendland 1.3i, 26); ibid. 98; Iamblichus, *Prot.* (Pistelli) 119–20; 124; *Comm. Math.* 36f., where it is ascribed to Archytas; Euseb. *Praep. Ev.* 2.9.3, 524a (Mras 2.24.10f.).

the third dimension, since – as mentioned above – the point is 1, the line 2, the plane 3, the solid 4. Aristotle therefore means: 'When the number of this scheme is translated into reality'.[40]

Whether other terms have or do not have mathematical meanings must be decided within the context. For instance the English phrase 'in addition' might mean 'when doing an addition sum', but as a rule it does not; similarly, the term 'function' has a precise mathematical meaning, but it is by no means its only one. We should therefore be aware that while αὔξησις and other words *may* be mathematical terms, their ordinary meanings cannot be ruled out in this text, which is not part of a mathematical treatise, even though it gives a mathematical illustration.

Socrates suggests invoking the Muses 'as Homer did', so we ought to expect a poetical application of mathematics, rather than a complicated calculation; we ought also to expect a surprise, a trick even, as the Muses are jesting and quizzing. Kurt Reidemeister considered this illustration a joke.[41] It is certainly a skit on Pythagoreanism, but it is also an integral part of the argument.

The problem is this: 'How then will our City be overthrown?' 'How does the first discord occur?' It is true that the other constitutions are also held to deteriorate through the intrusion of worse elements, but an explanation how this is possible is only required for the perfectly organized State. As Aristotle reports,[42] the Muses give a double answer; the first part is general: *It is indeed difficult for a city thus constituted to be overthrown; but as to all that comes into being destruction comes, so this constitution does not remain for all time but will be broken up.* The particular answer, which Aristotle calls 'the beginning (of the overthrow)' is *However wise those whom you have trained as leaders of the State may be, yet will they not always, through calculation and perception, manage to achieve good birth, or barrenness for your race, but they will err and produce offspring when it is not right.*

The leaders are the men responsible for the arranged 'marriages'.[43] O. Neugebauer[44] recognized this and also that our passage describes a process, but he misunderstood Socrates' intention. In a passage describing how yet another guess at Plato's number, related by 'wild artifices' to Babylonian texts, ruined the transcription and explanation of a whole set of tablets, Neugebauer wrote: 'In book VIII of his "Republic" Plato gives some cabbalistic rules as to how the guardians of his dictatorially ruled community should arrange for proper marriages'. But it is not a directive to the leaders; it is an explanation to the partners in the dialogue. Plato was not a cabbalist, whatever one may say of his followers.

The explanation is given in the form of a mathematical solution, very much condensed: it has only 4 of the 7 parts which Heath lists.[45] It starts with the Pythagorean view and shows that something has been overlooked.

1. *The enunciation of the problem* (προτάσις)

It must be explained why the leaders cannot avoid mistakes.

Plato's illustration is a series of right-angled triangles with rational sides. For each of these triangles a pair of sides (numbers) has to be found which fulfil the condition

[39] *De opif. mundi* 49.
[40] I.e. Plato's 'geometrical number', *Rep.* 546c fin.
[41] *Das exakte Denken der Griechen* (Hamburg, 1949), 17.
[42] Loc. cit. (n. 22).
[43] *Rep.* 459d f.
[44] *The Exact Sciences of Antiquity*[2] (Brown Univ. Press, 1957), 27.
[45] T. L. Heath, *The Legacy of Greece*, 103.

that the sum of their squares makes another perfect square: $a^2+b^2 = c^2$, where a, b and c are whole numbers. One of the sides is always odd, the other even (see Math. App. II). Even if the Pythagorean pair 'male – female'[46] is not found anywhere else in Plato[47] it is likely that the Muses here mean the two sides to represent men and women, 'yoked together' by the hypotenuse, which is here the bond, not (as in Plutarch[48]) the offspring; but Plutarch's number mysticism helps the understanding: to him 4 and 3 are the divine couple Isis and Osiris, 5 the divine child Horos; evidently a most successful pairing.

The first sentence, *There is for the divine offspring a cycle which a complete number comprehends*, gives the frame. The 'complete number' is likely to be the number of *Timaeus* 39d, the time which brings the eight spheres back to the positions relative to each other from which they started, the Great Year. Plato, very wisely, has not given this number; it cannot be calculated. So all guesses, whether looking to Babylon (as above or J. Adam[49]), or to astronomy (as e.g. Macrobius and Dupuis[50]), or to anywhere else, are equally legitimate and equally futile.

The following text deals with mankind and especially with the unavoidable errors of the leaders. In the first part, b4–8, there are two Pythagorean phrases which must be translated into ordinary mathematical terms; they can be found in ancient writers well before Proclus. M. Denkinger[51] believed that the four genitives ὁμοιούντων τε καὶ ἀνομοιούντων καὶ αὐξόντων καὶ φθινόντων were also mathematical terms; he accepted a meaning which A. Diès[52] had found in Proclus:[53] four rectangular solids, (*a*) with equal edges (cube), (*b*) with three different edges (little altar), (*c*) with a square base, the third edge being either longer (beam) or (*d*) shorter (plinth) than the side of the square. These are his 'four limits'. By a judicious distribution of three permutations of 3^4; 4^4; 5^4 as the edges of these four solids, and by multiplying the four volumes together, he builds up the number 12,960,000 in three different ways, besides finding it three more times in the following sentences. This seems excessive, but is neither surprising nor striking, as any desired number – except 1 and 2 – can be produced by multiplication and addition from 3, 4 and 5. The whole edifice rests solely on Proclus' assertion, since the other authors named – Theon, Nicomachus and Iamblichus – neither give Plato's words nor connect the solids in any way with the marriage number. But Proclus does not name four solids, but only three, treating the second pair as subordinate to the first: 'cubes' and 'solids with unequal sides, which are subdivided into beams and plinths'. Theon gives short descriptions which might support the construction,[54] but Nicomachus and Iamblichus rank the 'little altar' with solids like wedges and truncated pyramids; Nicomachus says further: 'others will call the same numbers "altar" using their own metaphor, for the altars of ancient style, particularly the Ionic, do not have the breadth equal to the depth, nor either of these equal to the length, *nor the base equal to the top*',[55] and Iamblichus: 'having

[46] E.g. Plutarch, *Quaest. Rom.* 264a; 288d; *De Is. et Os.* 373f.

[47] So according to M. Baltes, *Kommentar zu Timaios Lokros* (Leiden, 1972), 43; but Plato comes near to it in *Politicus* 262e.

[48] *De Is. et Os.* 373f.

[49] Op. cit. 41.

[50] Macrob. *Somn. Scip.* 2.11; Dupuis, op. cit. 371, 383.

[51] *REG* 63.

[52] *CRAI* 1933, 228–35, quoted after Denkinger, loc. cit. Cf. *Platon, Oeuvres*[6] (Paris, 1967), *Rep.* 8, p. 9 n. 1.

[53] *In Rep.* 2.36, 13f.

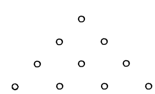
[54] Ed. Hiller, 9.41.

[55] *Arithm. Introd.* 17.6.

unequal plane faces, *unequal angles*, and unequal dimensions'.[56] And since the volume of such solids is not the product of their edges, Denkinger's calculation is wrong as well as improbable. Plato's words are to be taken in their ordinary sense; they recall the theme 'pairings', which may be congenial or uncongenial, thriving or declining. The next sentence continues these hints: there are the harmonies, there is the yoking together (a look into any dictionary will show how closely the stems *ζυγ-* and *ζευγ-*, which is the form Iamblichus uses when he repeats Plato's words,[57] are connected with the idea of joining in pairs) and there are the numbers of the triangle, Plutarch's *γαμήλιον διάγραμμα*, itself: 5 and 6 (the area) are Pythagorean numbers for marriage, being the sum and the product of the first even and the first odd number,[58] 'for the monad (is) not a number'.[59]

The first Pythagorean phrase is *δυνάμεναί τε καὶ δυναστευόμεναι*. For Ivor Thomas '*δυναστευόμεναι* is a *ἅπαξ λεγόμενον* and its meaning is uncertain...and perhaps we should not enquire too closely into what is more mystical than mathematical'.[60] In Plato *δυνάμενος* nearly always means 'able', 'capable'; in a few cases there is a connotation of political power, 'ruling',[61] and that, according to Alexander Aphrodisiensis,[62] has a bearing on our text: 'the Pythagoreans called the sides of the right-angled triangle *δυναστευόμεναι* ('the ruled'), the hypotenuse *δυνάμενος* ('the powerful')'. Iamblichus also calls the 5 *δυναμένη* (*γραμμή*).[63] The Muses must here be expected to use Pythagorean terms. M. Denkinger translates the terms 'par 5 et par 3 et 4' and 'dominées et dominatrice'.[64] This unusual expression has obscured for us the fact that the theme is triangles, but '4, 3, 5' and the 'oblong' (a double right-angled triangle) in the second part should be as plain to us as they were to the ancient commentators. The increases of hypotenuses and sides – we have no convenient adjectives for these nouns – are obtained by arranging the triangles in ascending order.

The other phrase, *τρεῖς ἀποστάσεις, τέτταρας δὲ ὅρους*, is found in Philo;[65] ten is formed by the first four numbers, 'which, laid down, have for boundaries the first, the second, the third, the fourth, and three distances, the one from 1 to 2, the second from 2 to 3, the third from 3 to 4'. This is beautifully Pythagorean: the perfect 10 as triangular number, laid down



Philo completes this:[66] 'indeed the infinite sequence of numbers is measured by it (the 10), since the boundaries framing it are four, 1 and 2 and 3 and 4, and the same boundaries produce the 100 out of the tens (10+20+30+40), similarly also the 1000 out of the hundreds etc.' Thus, where the modern mathematician gives algebraic definitions and proofs, the Greek 'just sees'

[56] *In Nic. Arithm.* 93–4.

[57] *Vita Pyth.* 27.130.

[58] Plut. *De E apud Delph.* 388c; *Quaest. Rom.* 264a; Arist. Quint. *De Mus.* 3.23; Theon 102.5, with a different explanation.

[59] E.g. Theon 102.1f.

[60] I. Thomas, *Greek Mathematical Works* (Loeb), 399 n.b.

[61] L. Brandwood, *Word Index to Plato* (Leeds, 1976) s.v.

[62] *In Metaph.* 56.19f., (Hayduck 75.31).

[63] Iamblich., loc. cit.

[64] *REG* 68 (1955), 38f.

[65] *De opif. mundi* 102, 47 (Cohn-Wendland 1.35.15.10f.). Cf. Plutarch, *Epit.* 1.3, in Diels, *Doxogr. Graeci*, p. 282.

[66] *De decalogo* 27, 4.274.

that the arithmetical rules must be valid throughout the infinite sequence of natural numbers even after we have run out of names for them. Neither Alexander Aphrodisiensis nor Philo can be suspected of inventing an explanation for Plato's text, because both are treating quite different subjects; we have here the authentic ancient meaning.

2. *The setting forth* (ἔκθεσις)

The passage in which Plato sets out the geometrical number which is to explain the leaders' errors (b 3–8) reads: *For the divine offspring there is a cycle which a complete number comprehends, but for the human* (*offspring a cycle which a different number comprehends*) *in which first* (*number*) *right-angled triangles in ascending order, which have incorporated the sequence of natural numbers, as defined by the ten, render everything, of the assimilating and the dissimilating and of the growing and the decaying, concordant and rational.*

3. *The construction* (κατασκευή).

In this passage (to c 5) Plato delimits the series more precisely, giving its first term and the rule which governs it. There are no hidden mathematical terms, but two words need some explanation: (i) The definition of πυθμήν in Liddell-Scott-Jones s.v. III, 'base of a series, i.e. the lowest number [better "lowest term"] possessing a given property', is good, but not the translation of *Rep.* 546c in the second example ('the first couple of numbers giving the ratio...'): it is in Nicomachus 216 that the series consists of ratios; here the first term is what Aristides Quintilianus and Alexander Aphrodisiensis call the first Pythagorean triangle.[67]

(ii) The harmonies are, as Aristides Quintilianus says, musical harmonies (the only meaning found in Plato) and Plato names them, 'the epitriton' and the 'dia pente' or ἡμιόλιον. These were the first two chords for which the Pythagoreans – the legend says Pythagoras himself – had found the numerical values, the comparative lengths of the string, and their ratio 3/2:4/3 gave the 'epogdoon' (9/8),[68] the whole tone, which in turn furnished the numerical values of the whole scale. The harmonies do not enter into the calculation; they are mentioned in praise of the famous triangle. This was often praised: Philo called it ἀρχὴ ποιοτήτον and τῆς τῶν ὅλων γενέσεως ἀρχή,[69] Plutarch 'the most beautiful of all triangles'.[70] This is Pythagorean; Plato's 'most beautiful of all triangles' is half an equilateral,[71] whose sides are not commensurable and whose angles are 90°, 60°, 30°. Plato limits the series to 100 terms, i.e. the numbers 3 to 102 have to be paired if possible. At each step all three numbers (sides) increase; the first number is to be squared; its length forms with the second number – tied to a pair by μέν...δέ – an oblong, i.e. a double right-angled triangle, with a rational diameter. There are potentially 100 such diameters, but this 100 is – again by μέν...δέ – tied to the 3^3 at the end of the next passage in which the result is discussed.

The lowest term of these (triangles), *the 4/3 yoked together by the 5, produces two harmonies, three times increased* (*1*) *in the line multiplied by itself – one hundred such –*,

[67] Arist. Quint. *De Mus.* 3.23; Alex. Aphrod., *In Metaph.* 56.19f.

[68] Aristoxenux, *Harmonica*, ed. H. S. Macran 1.21, 20; 1154; 46, 1; Plato, *Tim.* 36; Diels, *Vorsokrat.* 1.409f.; Philolaos frg. 6; Philo, *De opif. mundi* 96 fin.; Arist. Quint. loc. cit.; Macrob. *Somn. Scip.* 2.1.

[69] *De opif. mundi* 97, 1.33; *Vita Mosis* 2.79 (C.-W. 4.219).

[70] *De Is. et Os.* 373f.

[71] *Tim.* 54a f.

(2) in the rectangle which has one side equal to this (line), (3) in the potentially 100 numbers (starting) from the rational diameters' five.

The first sides are the sequence of natural numbers; no indication is given how the other side and the hypotenuse are to be found, for 'the Socrates of the *Republic* presupposes this knowledge in the partners of his dialogue; he only interprets the well-known mathematics...for his purpose'.[72] Fortunately, Hero supplies a suitable formula under the name of Pythagoras:[73] the Pythagorean method from an odd number' – in algebraic terms: if a is an odd number, $b = (a^2-1)/2$ is the other side, and $c = (a^2+1)/2$ the hypotenuse. This gives the triads 3, 4, 5; 5, 12, 13; 7, 24, 25; 9, 40, 41; etc., in which the three numbers are each increasing at each step, they are found from the square of the first number, which is at the same time a side of the right-angled triangle (or of the rectangle), and all three sides are whole numbers.

Tables of Pythagorean triangles had existed in Babylon long before Plato's time,[74] but the Greeks, being mathematicians, not just technologists, formulated the laws.

Plato also presupposes these further items of knowledge: the restriction 'up to 102' applies to the given side only – the other side and the hypotenuse growing much faster – and the problem is to find Pythagorean triangles for as many of these hundred numbers as will fulfil the conditions; also, all triads have to be at their lowest terms, otherwise great numbers of identical triangles, only drawn on different scales, would result.[75]

So far, everything has been Pythagorean and tidy. The 3 belongs to the 4 and the 4 to the 3 exclusively, and the same seems to apply to all the other couples; there is no room for errors. The sting comes in the tail, in the last passage, in which the result is discussed. As T. L. Heath[76] explains, 'it often happens that a solution is not possible unless the particular data are such as to satisfy certain conditions; in this case there is...the statement of the conditions or statement of limits of possibility'. Pythagoras' formula has paired all the odd numbers, fifty of them, and, incidentally, six even numbers within the given range. These are the quadruples of the first six triangular numbers (see Appendix, section 3), viz. 4, 12, 24, 40, 60 and 84. In the hundred, 56 numbers have been matched, 44 not; but as the count in the last passage includes 1 and 2 – it is usual even today to repeat that 1 and 2 make no Pythagorean triangles – the 'missing' numbers amount to 46. But the last sentence says three cubed, i.e. 27, a discrepancy of nineteen. This is due to a second formula, which Hero gives under the name of Plato:[77] 'According to Plato from an even number'. If this tradition in Hero has a sound historical base, it would provide the 'recent discovery' which J. Dupuis postulated.[78] The formula was of course well known to Socrates' audience; in algebraic terms it runs as follows: if b is an even number, $a = (b/2)^2 - 1$ is the other side, $c = (b/2)^2 + 1$ is the hypotenuse. For 'even odd' numbers (i.e. those with only one factor 2)[79] this formula gives three even numbers; when these are

[72] K. Reidemeister, *Das exakte Denken der Griechen*, 50 (commenting on a different passage).

[73] Hero Alex. *Geometr.* 8.1; also found in Proclus, *Eucl.* (Friedlein) 424.

[74] Neugebauer, *The Exact Sciences in Antiquity* 36f. The interest in and knowledge of 'Pythagorean' numbers was not confined to the Near East. Unfortunately I cannot find the source, published some years ago, for the following information about an Orkney Henge: 'The [inaccurate] circle was in fact an ellipse, described from a triangle with sides in the ratio 5:12:13, and with diameters in integral numbers of megalithic yards.'

[75] Cf. Plato, *Tim.* 54, where the similar right-angled isosceles triangles are all 'of the same nature'.

[76] Op. cit. 103.

[77] *Geometr.* 9.1; repeated by Proclus, *Eucl.* (Friedlein) 467.

[78] *Théon de Smyrne*, 370. See T. L. Heath, *The Legacy of Greece* 117.

[79] 'Artio-perissoi' (see any ancient *Arithmetic*), Plato, *Parmen.* 143e.

divided by 2, they reduce to the triad which has already been found for the odd factor, i.e. they produce no (new) triangles; these are the 'twos which cannot be expressed';[80] of the remaining twenty-five even numbers, six had already been found, so that nineteen numbers are added.

Taken by itself, the new, Platonic, series also has the three numbers increasing at each step; but when one uses the sequences of natural numbers, as the Muses require, the result is as follows:

1 no triangle	7, 24, 25
2 no triangle	8, 15, 17
3, 4, 5	9, 40, 41
4, 3, 5	10, 24, 26 (same as 5, 12, 13)
5, 12, 13	11, 60, 61
6, 8, 10 (discarded, since same as 3, 4, 5)	12, 35, 37 etc.

The beautiful order is destroyed: there are no longer three increases at each step.

But what is worse, the apparent certainty of Pythagoras' pairing no longer applies, as all twenty-five even numbers – except the 4 – produce new, different triangles by Plato's formula, the quadruple triangular numbers now appear with different odd numbers, and several odd numbers (only those up to 102 matter) have different even partners; nor is there any rule by which it could be decided whether e.g. the 12 should be paired with the 5 or with the 35, or whether the 15 should have 8 as its partner or 122. There is now considerable scope for unavoidable errors.

4. *The statement of limits* (διορισμός)

Missing in each (of these pairs[81]) *are the one, and the twos which cannot be expressed, of the whole one hundred* (numbers) *in fact three cubed* = 27 numbers. *This whole geometrical number*, Plato continues, *is decisive for this, the better and the worse births*, i.e. for the errors which the leaders cannot avoid. It is usually considered that the certainty which we cannot obtain about our material surroundings can be achieved in mathematics. If it is then impossible to decide which of two pairs of numbers is to be preferred, it is obvious that the situation must be worse in real life, 'when the number of this scheme is translated into reality' as Aristotle said.

The following sentences do not reflect on the leaders' competence, but state a fact, that this is the way the world is made: *whenever your Guardians, not knowing these things, bring together brides with bridegrooms inopportunely, there will be children which are neither of good disposition nor successful.*

This kind of scheme shows how much more sensitive the Greek mathematicians were to the quality of numbers than latter generations, with the exception of those whom David Hilbert called 'the real mathematicians',[82] the theorists of number. It also shows that those minute distinctions between 'all even', 'odd even' and 'even odd' numbers and the like were of real importance.

Other 'numbers' occurring in this passage, 'triangular', 'odd' – which make squares –, 'even' – which make *heteromekeis* (see Appendix, section 4) –, might also be called 'geometrical'; they too are not this or that definite number, but types with

[80] Nic. Geras. 2.17: Even numbers partake of the nature of the 2.
[81] Cf. Plato, *Legg.* 759d.
[82] In a lecture.

certain qualities, like the Pythagorean numbers, which the Greeks, having no algebra, defined in geometrical terms. 'Polygon numbers' can be found in any ancient *Arithmetic*: Theon[83] appends to them, under the title 'on side and diameter numbers', the above mentioned series of figures which converge towards $\sqrt{2}$. The Pythagorean series does not appear in these later *Arithmetics*; but the references to 'the first triangle' or 'the first diameter' show that it was not altogether forgotten.

Euclid gives a formula[84] by which a given number can be expressed as the difference of two squares by pairs of its factors (see Appendix, section 5). This can be easily transformed into a general formula for finding Pythagorean numbers, by choosing a square as this number. Nicomachus of Gerasa promised: 'These matters will receive their proper elucidation in the Commentary on Plato, with reference to the passage of the so-called marriage number in the *Republic*, introduced in the person of the Muses'.[85] This work is not extant. As Nicomachus had defined the 'odd', 'all even', 'even odd' and 'odd even' numbers in the usual way and – more unusually – had discussed factors, with a special paragraph on factors of squares, showing the different ways these factors can be paired, it is possible that he meant to or actually did use this general formula to show that the position was even worse than Plato had realized, as the number of different pairings rises with the number of different factors.

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APPENDIX

1. *Approximation to $\sqrt{2}$ by rational ratios*

In a sequence of isosceles triangles, let the side be s, and the base (the diameter) be b:

$$s_1 = b_1 = 1; \quad s_{n+1} = s_n + b_n \quad \text{and} \quad b_{n+1} = b_n + 2s_n.$$

Therefore, $$2s_{n+1}^2 - b_{n+1}^2 = 2(s_n + b_n)^2 - (b_n + 2s_n)^2 = -(2s_n^2 - b_n^2).$$

i.e. the difference between the sum of the squares on the sides and the square on the base is equal to the negative value for the preceding term. As the value of the first term is 1, the difference is always plus and minus 1 in turns, therefore as the numbers get larger, the ratio $b_n : s_n$ is approaching – but never reaching –, the value of $\sqrt{2}$ from both sides. The angle at the apex, meanwhile, varies between 60° and somewhat over 97°, approaching from both sides – but never reaching – 90°.

Numerical values

n	s_n	b_n	b_n/s_n	In decimals
1	1	1	1	1
2	2	3	3/2	1·5
3	5	7	7/5	1·4
4	12	17	17/12	$1{\cdot}416\bar{6}\ldots$
5	29	41	41/29	1·4138...
6	70	99	99/70	1·4142...
7	169	239	239/169	1·41420...

[83] Op. cit. 43f. [84] 10.28a. [85] 2.24, 10 fin., transl. by M. L. D. Ooge.

Steps 6 and 7 already provide $\sqrt{2} = 1.4142$ correct to five places. The numbers in this series are carefully produced, for twice a perfect square is by no means usually within 1 of another perfect square. One would like to know what Proclus and his followers would suggest as 'rational diameter' for $2 \times 4^2 = 32$ or $2 \times 6^2 = 72$.

2. *In a Pythagorean triangle, one side is even, the other side odd, the hypotenuse always odd*

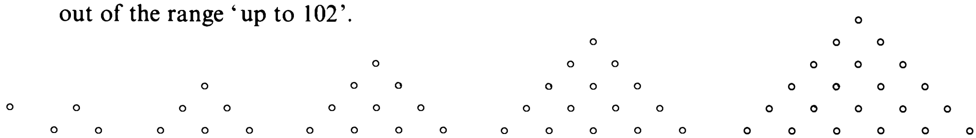
(i) Assume both sides are even. It follows that the square on the hypotenuse and therefore the hypotenuse are also even, and the numbers are not at their lowest terms. They must be divided by 2 till at least one side (or the hypotenuse) is odd, which makes the hypotenuse (or one side) also odd.

(ii) If both sides are odd, they can be written $2m+1$ and $2n+1$, so the sum of the squares on these sides is $4(m^2+n^2+m+n)+2$, which is an even number but not divisible by 4 and therefore not a perfect square, and the triangle is not a Pythagorean triangle.

It follows that one side must be odd the other even, and that makes the hypotenuse odd.

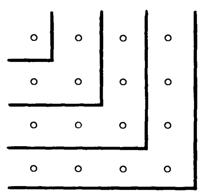
3. *The even sides of Pythagorean triangles found by Pythagoras' formula (for an odd number) are quadruples of triangular numbers*

The odd side $a \geqslant 3$ can be written $2n+1$. The even side $(a^2-1)/2$ then becomes $[(2n+1)^2-1]/2 = 4[(n+1)n/2]$; this is 4 times the sum of an arithmetical progression, in this case the sum of the natural numbers from 1 to n, which form a triangular number. The six numbers in question are $4 \times 1 = 4$; $4 \times 3 = 12$; $4 \times 6 = 24$; $4 \times 10 = 40$; $4 \times 15 = 60$; $4 \times 21 = 84$. The next following, $4 \times 28 = 112$, is already out of the range 'up to 102'.

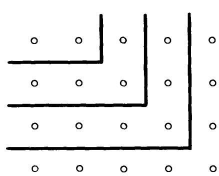


4. *Odd numbers make squares*

Algebraically: $\Sigma_{m-1}^{n} [(2m-1) = (1+2n-1)n/2] = n^2$

Graece the same represents  etc.

Even numbers form heterometeis, i.e. oblongs, the sides of which differ by 1.
Algebraically: $\Sigma_{m-1}^{n} 2m = [(2n+2)n/2] = (n+1)n$.

Graece the same represents  etc.

5. The general formula

This is arrived at by choosing a square as the given number.

$$a^2 = rs = \left(\frac{r+s}{2}\right)^2 - \left(\frac{r-s}{2}\right)^2.$$

The factors r and s must be chosen so that $r > s$, they are either both odd or both even and have no common factor greater than 2. E.g.

$$20^2 = 50 \times 8 = \left(\frac{50+8}{2}\right)^2 - \left(\frac{50-8}{2}\right)^2 = 29^2 - 21^2.$$

The resulting triad 20, 21, 29 provides a new partner for 20 as well as for 21. 60 and 84 have four solutions each altogether, though only two of the four are new ones.

'The twos that cannot be expressed' cannot be expressed by this formula either: each of the factors of their squares has only one factor 2, so that half the sum and half the difference of their factors are even numbers, and therefore both sides and the hypotenuse are even.